Diophantine Approximation and irrationality of zeta values

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1 Introduction

In this article, we aim to provide the reader with a taste of Diophantine approximation, closely following the book Exploring the Number Jungle by Edward B. Burger, and some occasional references to the classical book An Introduction to the Theory of Numbers by Hardy and Wright.[1]

Firstly, what is Diophantine Approximation, and why do we even bother with it? ....

2 Farey Sequences

2.1 Properties of the Farey Sequences

Here, we uncover the arithmetic structure of the Farey Sequences. Most of the observations here were made by J.Farey in 1816. However, he gave no proof. [HW09, p.44] Cauchy, however, immediately provided the proofs after seeing Farey's statement.

Definition 2.1. A *Farey Sequence of order N* is the ordered list of all reduced fractions between 0 and 1 having denominators not exceeding N. We denote this sequence by $\mathfrak{F}_\mathfrak{n}$

So for example,

$$\mathfrak{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$\mathfrak{F}_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$\mathfrak{F}_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}.$$

We take a slight detour, first considering the size of $\mathfrak{F}_N$, before considering its arithmetical structure.

Let $\#(\mathfrak{F}_N)$ denote the number of elements in $\mathfrak{F}_N$. For example,
$\#(\mathfrak{F}_1) = 2, \#(\mathfrak{F}_2) = 3, \#(\mathfrak{F}_3) = 5, \#(\mathfrak{F}_4) = 7$ and so on. Looking at the first few values for $\#(\mathfrak{F}_N)$, one might conjecture that $\#(\mathfrak{F}_N)$ is always prime. But this is in fact not true. To disprove this conjecture, we first need a few lemmas.

Lemma 2.2. *For a positive integer $N \geq 2$, the number of elements in $\mathfrak{F}_N$ having denominator N is equal to $\varphi(N)$, where $\varphi(N)$ is the Euler totient function.*

[1] A slight disclaimer: since the book is more of an exercise book, most of the proofs, unless stated otherwise have been done by me along with some help from my supervisor. Hence, the proofs may not be the most elegant, and all errors are most certainly mine.

Proof. By Definition 2.1, the elements with the denominator N are precisely the fractions that have numerators which are coprime to N, and the number of them is given by $\varphi(N)$. $\square$

With this, we can derive a formula for $\#(\mathfrak{F}_N)$.

Proposition 2.3. *For any $N \geq 2$,*

$$\#(\mathfrak{F}_N) = 1 + \sum_{m=1}^{N} \varphi(m).$$

Proof. The statement is true for $N = 1$. Assuming that it is true for $N = n$, the number of elements that are in $\mathfrak{F}_{n+1}$ is the sum of the number of elements in $\mathfrak{F}_n$ and the new elements with numerator coprime to $n + 1$ and denominator $n + 1$, of which there are $\varphi(n + 1)$ of them. $\square$

Using this formula, we check that $\#(\mathfrak{F}_{10}) = 33$, which disproves our conjecture.

Remark. While there does not exist a simple, closed formula for the formula in the previous proposition, the simple function $f(N) = \frac{3}{\pi^2}N^2$ approximates the formula surprisingly well for sufficiently large N. For example, $\#(\mathfrak{F}_{10}) = 33$ while $\frac{300}{\pi^2} \approx 30.4$. [2]

We now go back to uncovering the underlying arithmetical structure of the Farey Sequences. From here onwards, we shall assume that all rationals discussed are positive, along with the natural numbers p, q, r, s, a, b, x, y

Lemma 2.4. *Let $\frac{p}{q} < \frac{r}{s}$ be two rational numbers satisfying $ps - rq = -1$. Then for all positive integers λ and μ, we have*

$$\frac{p}{q} \leq \frac{\lambda p + \mu r}{\lambda q + \mu s} \leq \frac{r}{s}.$$

Proof. We have that the statement is equivalent to:

$$\begin{aligned}
& \frac{p}{q} \leq \frac{\lambda p + \mu r}{\lambda q + \mu s} \leq \frac{r}{s} \\
\iff \quad & \frac{p}{q} - \frac{r}{s} \leq \frac{\lambda p + \mu r}{\lambda q + \mu s} - \frac{r}{s} \leq 0 \\
\iff \quad & \frac{ps - rq}{qs} \leq \frac{\lambda ps + \mu rs - \lambda rq - \mu rs}{\lambda qs + \mu s^2} \leq 0 \\
\iff \quad & \frac{-1}{qs} \leq \frac{-\lambda}{\lambda qs + \mu s^2} \leq 0 \\
\iff \quad & \frac{-1}{qs} \leq \frac{-1}{qs + \frac{\mu}{\lambda} s^2} \leq 0,
\end{aligned}$$

which is true, since all λ, μ, q, s are positive. $\square$

It what follows, it may be useful to note that the hypothesis in the previous Lemma could be written as

$$\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = -1$$

Lemma 2.5. *Let $\frac{p}{q} < \frac{r}{s}$ be two rational numbers satisfying $ps - rq = -1$. Suppose that $\frac{a}{b}$ is a rational number satisfying $\frac{p}{q} \leq \frac{a}{b} \leq \frac{r}{s}$. Then there exists nonnegative integers λ and μ such that*

$$a = \lambda p + \mu r \quad \textit{and} \quad b = \lambda q + \mu s.$$

[2]The reader might recognise this particular function to be the average order of the Euler totient function.

Proof. We solve directly for λ and μ. We have that:

$$a = \lambda p + \mu r$$
$$b = \lambda q + \mu s,$$

Multiplying the first equation by s, and the second equation by r, then solving for λ:

$$as = \lambda ps + \mu rs \quad br = \lambda qs + \mu rs$$
$$as - br = \lambda ps - \lambda qr$$
$$\lambda = \frac{as - br}{ps - qr}.$$

Now since $ps - rq = -1$, it follows that $\lambda = br - as$, and similarly, solving for μ we get $\mu = bp - aq$. Since all a, b, r, s are natural numbers, we are done. □

Combining these two lemmas, we have the following.

Theorem 2.6. *Let $\frac{p}{q} < \frac{r}{s}$ be two rational numbers satisfying $ps - rq = -1$. Then a rational number $\frac{a}{b}$ is an element of the interval $[\frac{p}{q}, \frac{r}{s}]$ if and only if $a = \lambda p + \mu r$ and $b = \lambda q + \mu s$ for some non-negative integers λ and μ.*

Proof. Follows from Lemma 2.4 and Lemma 2.5. □

Before proceeding further, we prove the following propositions, whose results come in handy when proving further results.

Proposition 2.7. *Suppose that $ps - rq = -1$. If $\gcd(\lambda, \mu) = 1$, all fractions of the form*

$$\frac{H}{K} = \frac{\lambda p + \mu r}{\lambda q + \mu s}$$

for non-negative integers λ and μ are in lowest terms.

Proof. We first note that $\gcd(p, q) = 1$, and $\gcd(r, s) = 1$. Now,

$$H = \lambda p + \mu r \quad \text{and} \quad K = \lambda q + \mu s.$$

After multiplying the first equation by q and the second equation by p, we get

$$Hq = \lambda pq + \mu rq$$
$$Kp = \lambda pq + \mu sp.$$

Subtracting the first equation from the second and using the condition $ps - rq = -1$,

$$Hq - Kp = \mu.$$

Similarly, we can obtain :

$$Hs - Kr = -\lambda.$$

Since $\gcd(\lambda, \mu) = 1$, the only common divisor between K and H is 1, from which the claim follows. □

Proposition 2.8. *No two consecutive elements in $\mathfrak{F}_n$ can have the same denominator for $n > 1$.*

Proof. If $q > 1$ and $\frac{r}{q}$ succeeds $\frac{p}{q}$ in $\mathfrak{F}_n$, then we have that $p + 1 \leq r \leq q$. But

$$\frac{p}{q} < \frac{p}{q-1} < \frac{p+1}{q} < \frac{r}{q}.$$

So $\frac{p}{q-1}$ comes in between $\frac{p}{q}$ and $\frac{r}{q}$, which is a contradiction. □

Proposition 2.9. *If $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive fractions in $\mathfrak{F}_N$, with $b, d \geq 2$, then*

$$\frac{1}{bd} < \frac{1}{N}.$$

Proof. We note that $b + d > N$, otherwise the two fractions cannot possibly be consecutive. WLOG, let $b > d$, so

$$bd \geq 2b > b + d > N.$$

□

We are now ready to prove the key properties of the Farey Sequences.

Theorem 2.10. *Let $\frac{p}{q} < \frac{r}{s}$ be any two consecutive fractions in $\mathfrak{F}_n$. Then*

$$\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = -1$$

Proof. We proceed by induction on n. The base case when $n = 1$ is true. Assume that the statement holds for $\mathfrak{F}_{n-1}$. Let $\frac{p}{q} < \frac{r}{s}$ be two consecutive elements in $\mathfrak{F}_{n-1}$. Now suppose that $\frac{a}{b}$ separates $\frac{p}{q}$ and $\frac{r}{s}$ in $\mathfrak{F}_n$. Using Theorem 2.3, we infer that

$$\frac{a}{b} = \frac{\lambda p + \mu r}{\lambda q + \mu s},$$

for some non-negative integer λ and μ. We now claim that $\lambda = 1$ and $\mu = 1$, and that $b = q + s$. Suppose not, since $\frac{a}{b}$ is in $\mathfrak{F}_n$ but not in $\mathfrak{F}_{n-1}$, we must have $b = N$. So $N = \lambda q + \mu s$, for some λ, μ (either one) ≥ 2. Define a new fraction $\frac{\tilde{a}}{\tilde{b}}$ with $\tilde{a} = p + s$ and $\tilde{b} = q + s$. Again by Theorem 2.3 this fraction is in between $\frac{p}{q} < \frac{r}{s}$. But since $q + s < \lambda q + \mu s < N$, this fraction will be in $\mathfrak{F}_{n-1}$, contradicting our assumption that $\frac{p}{q}$ and $\frac{r}{s}$ are consecutive elements in $\mathfrak{F}_{n-1}$. □

Theorem 2.11. *The fractions that belong to $\mathfrak{F}_n$ but not $\mathfrak{F}_{n-1}$ are mediants of two consecutive elements of $\mathfrak{F}_n$.*

Proof. Let $\frac{a}{b}$ be the fraction in question. Let $\frac{p}{q}, \frac{r}{s}$ be such that $\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$ are consecutive elements in $\mathfrak{F}_n$. Since $\frac{a}{b}$ is consecutive to both $\frac{p}{q}$ and $\frac{r}{s}$ in $\mathfrak{F}_n$, we have that $n = b \neq r, s$, and since $\frac{p}{q}$ and $\frac{r}{s}$ are consecutive in $\mathfrak{F}_{n-1}$, ($\frac{a}{b}$ is not an element of $\mathfrak{F}_{n-1}$) by Proposition 2.8, we have that $b \neq r \neq s$. Also, from Theorem 2.6, we have that $\frac{a}{b}$ is of the form

$$\frac{a}{b} = \frac{\lambda p + \mu s}{\lambda q + \mu s}.$$

A similar argument to the proof of Theorem 2.10 shows that we must have $\lambda = \mu = 1$, and we are done. □

Remark. The converse of both previous theorems are not true. For Theorem 2.10, consider the consecutive elements of $\mathfrak{F}_5 : \frac{1}{3}, \frac{2}{5}, \frac{1}{2}$. We have that

$$\det \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = -1,$$

but they are not consecutive elements.

For Theorem 2.11 consider the consecutive elements in$\mathfrak{F}_3 : \frac{1}{3}, \frac{1}{2}$. We have that $\frac{1+1=2}{3+2=5}$ is not in $\mathfrak{F}_4$.

With striking similarity of the proof strategies for the previous two theorems, one might guess that Theorem 2.10 and Theorem 2.11 are equivalent. Indeed, they are. (I omit the proof for now, and will decide whether to add it in later, but it is in Hardy and Wright)

2.2 Ford Circles

We take a slight detour from analysing the elements of the Farey Sequences in an arithmetic sense, and take a more geometrical approach, namely via Ford Circles. They are a different representation of the Farey Sequences and can give some insight to how well we can approximate arbitrary real numbers.

Definition 2.12. For a rational number $\frac{p}{q}$, we define the circle $C(\frac{p}{q})$ in the complex plane having centre at $\frac{p}{q} + \frac{1}{2q^2}i$ and radius $\frac{1}{2q^2}$. Such circles are called *Ford Circles.*

We present a sketch of the Ford Circles for the rationals in each set: $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \mathfrak{F}_5$ below:

(Figure out how to insert someday)

We see that the Ford Circles pack together peculiarly; one might make a few observations about them, and we provide proofs of those observations.

Theorem 2.13. *The intersection of the interiors of any two distinct Ford Circles is empty.*

Proof. It suffices to show that the distance between any two circles is greater than the sum of their radii. WLOG: we assume that $\frac{p}{q} < \frac{r}{s}$. We aim to show that

$$\sqrt{(\frac{p}{q} - \frac{r}{s})^2 + (\frac{1}{2q^2} - \frac{1}{2s^2})^2} \geq \frac{1}{2q^2} + \frac{1}{2s^2}.$$

Which is equivalent to

$$\begin{aligned} &(\frac{p}{q} - \frac{r}{s})^2 + (\frac{1}{2q^2} - \frac{1}{2s^2})^2 \geq (\frac{1}{2q^2} + \frac{1}{2s^2})^2 \\ \iff &(\frac{p}{q} - \frac{r}{s})^2 \geq \frac{1}{s^2q^2} \\ \iff &(ps - rq)^2 \geq 1, \end{aligned}$$

and this is true since all p, q, r, s are non-negative integers. □

Theorem 2.14. *Two Ford Circles, $C(\frac{p}{q})$ and $C(\frac{r}{s})$, are tangent if and only if $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent elements in $\mathfrak{F}_n$ for some N.*

Proof. We start with right implies left:

Suppose $\frac{p}{q}$ and $\frac{r}{s}$ are consecutive elements in $\mathfrak{F}_n$, then we have that $ps - rq = -1$, and taking this into Theorem 2.13 we are done.

Left implies right:

Since the circles are tangent to each other, by the previous theorem we deduce that $ps - rq = -1$. Let $N = max\{q, s\}$. It follows that both fractions are elements of $\mathfrak{F}_N$. Suppose $\frac{p}{q} < \frac{r}{s}$ are not adjacent. So there exists a fraction, say, $\frac{a}{b}$, also an element of $\mathfrak{F}_N$ seperating them. From Theorem 2.6, we have

$$\frac{a}{b} = \frac{\lambda p + \mu r}{\lambda q + \mu s},$$

for some nonnegative integers λ, μ. Since $N = max\{q, s\}$, we consider two cases for b.

1. $b = N$

   WLOG, let $N = s = b$, it follows that we must have $\mu = 0$. So $\frac{a}{b} = \frac{\lambda p}{\lambda q} = \frac{p}{q}$, which is a contradiction.

2. $b < N$

   Again, WLOG, suppose that $N = s$. Just as before, we must have $\mu = 0$. Notice that since $\frac{a}{b}$ is defined, we cannot never have $0\lambda = \mu = 0$. A similar argument to the previous case completes the proof.

□

Theorem 2.15. *Suppose that the Ford Circles $C(\frac{p}{q})$ and $C(\frac{r}{s})$ are tangent to each other. Let $\mathcal{C}$ be the circle that is tangent to $C(\frac{p}{q})$ and $C(\frac{r}{s})$ and the real axis. Then $\mathcal{C} = C(\frac{p+r}{q+s})$.*

Proof. □

In fact, we can use the Ford Circles to find excellent rational approximations to $\alpha \in \mathbb{R}$. One way we can do this is by starting with the complex plane together with all the Ford Circles tangent to the real axis. Then we draw a ray perpendicular to the real axis starting from α approaching to $+\infty$. This ray w ill intersect an infinite sequence of shrinking Ford Circles. The points of tangency of those circles to the x-axis form a sequence of rational numbers approaching α, all having relatively small height [3]. The upper bound of the error will be $\frac{1}{2q^2}$, the radius of the Ford Circles.

With that in mind, we proceed to the main results in Diophantine Approximation.

3 Discoveries of Dirichlet and Hurwitz

3.1 Dirichlet's Theorem

Before proving Dirichlet's Theorem, one of the fundamental theorems in Diophantine approximation, we will need the Pigeonhole Principle.

Theorem 3.1 (Pigeonhole Principle)**.** *Suppose we have N boxes and $N+1$ objects. If we were to place the objects in the boxes, then no matter how we placed the objects, there must exist a box containing more than one object.*

Proof. We present two proofs, the first by contradiction and the second by induction.

1. Proof by contradiction: Suppose that there does not exist a box that contains more than one object. Then we have a total of N boxes, each containing only one object, adding up to N objects, which contradicts the fact that we have $N+1$ objects.
2. Proof by induction: The base case is trivial, when we have two objects but only one box, we must have the box contains more than one object. Now we suppose that the result is true for $N+1$, objects and N boxes, when we have $N+2$ objects and $N+1$ boxes, we can assign one and only one object to a single box and we are back in the $N+1$ objects case. Hence, we are done.

□

As a minor digression, we give an application of Theorem 3.1, demonstrating the usefulness of this intuitively obvious theorem.

Theorem 3.2. *Given any set of N positive integers, there exists a subset whose sum is divisible by N.*

Proof. Let the set of N integers be denoted by $\{a_1, a_2, \dots, a_N\}$. If any one of the integers is a multiple of N, then we are done. So suppose not. Define the sum

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &\vdots \\ S_i &= a_1 + a_2 + \cdots + a_i \\ &\vdots \\ S_N &= a_1 + a_2 + \cdots + a_N. \end{aligned}$$

Again, if any one of the S_i is a multiple of N, then we are done. So suppose that none of the S_i is a multiple of N. Then $S_i \equiv 1, 2, \dots, N-1 \pmod{N}$. Now note that we have N sums but only $N-1$

[3]The height of an integer is defined by $h(\frac{p}{q}) = \max\{|p|, |q|\}$. We can think of this 'height' to be a measure of complexity.

residues modulo N, so by Theorem 3.1 there must exist $i.j$ such that $S_i \equiv S_j \pmod{N}$. WLOG, let $i > j$, we take the subset formed by the elements of $S_i - S_j : \{a_j, a_{j+1}, \dots, a_i\}$ and we are done. □

We now prove the fundamental result of Dirichlet from 1812.

Theorem 3.3 (Dirichlet, 1812)**.** *Let α be a real number and Q be a positive integer. Then there exists a rational number $\frac{p}{q}$ such that $0 < q \leq Q$, and*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q(Q+1)}.$$

Proof. We note that the inequality is equivalent to

$$|\alpha q - p| \leq \frac{1}{(Q+1)}. \tag{1}$$

Hence, now our objective is to find integers p, q such that they satisfy (1). We partition the interval $[0,1]$ into $Q+1$ equal intervals. Write $i\alpha = \lfloor i\alpha \rfloor + \{i\alpha\}$,[4] where $i \in \{0, 1, \dots, Q\}$. Consider the values $\{i\alpha\}$, and the value 1. In total we have $Q+2$ values in the interval $[0,1]$, but only $Q+1$ partitions. So by Theorem 3.1 there exist two values amongst the set containing $\{i\alpha\}$ and 1 that are in the same interval. We break this down into two cases. The first case is where both values are from the set of $\{i\alpha.\}$. Then WLOG, suppose $0 \leq j < i \leq Q$, it follows that

$$\big|\{i\alpha\} - \{j\alpha\}\big| \leq \frac{1}{(Q+1)}.$$

Using the definition of $i\alpha$, we are done. Now suppose that we have the two values lying in the same interval being 1 and $\{i\alpha\}$ for some i. Note that we cannot have $i = 0$. So

$$\big|\{i\alpha\} - 1\big| \leq \frac{1}{(Q+1)}.$$

Hence

$$\big|i\alpha - \lfloor i\alpha \rfloor - 1\big| \leq \frac{1}{(Q+1)},$$

which concludes the proof. □

Remark. For certain α's, Theorem 3.3 cannot be improved. For example, for $\alpha = \frac{a}{b}$ for some integers a, b, we can take $p = \frac{aQ}{b} - \frac{1}{Q+1}$ in (1) and both bounds in Theorem 3.3 will be sharp.

We now deduce an immediate consequence of Dirichlet's Theorem.

Theorem 3.4. *A real number α is irrational if and only if there exist infinitely many rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Proof. [HW09, p.201-202] We start with the 'only if' direction, suppose that there are only finitely many rationals, say n of them, indexed by $\frac{p_i}{q_i}$ that satisfy the statement. Then since α is irrational, we have that $\alpha - \frac{p_i}{q_i} \neq 0$ for each i. So there exists a natural number Q such that

$$\left|\alpha - \frac{p_i}{q_i}\right| > \frac{1}{Q+1}.$$

By Theorem 3.3, for this value of Q, there exists another rational, $\frac{p}{q}$ with

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q(Q+1)} \leq \frac{1}{Q+1}.$$

[4] We use {x} to denote the fractional part of x, and is in the range [0,1)

For this rational $\frac{p}{q}$, we also have

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Hence, we found another rational not in our list of $\frac{p_i}{q_i}$ that satisfies the statement, a contradiction.

For the 'if' direction, we prove the contrapositive: If α is rational, then there exists only finitely many rationals $\frac{p}{q}$ with

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

WLOG, let $\alpha \in [0,1]$,. Since if $\alpha > 1$, we can always write $\lfloor \alpha \rfloor = k = \frac{kq}{q}$, and use $\tilde{p} = p + kq$ instead of p in the inequality. Write $\alpha = \frac{a}{b}$, then

$$\begin{aligned} \left|\frac{a}{b} - \frac{p}{q}\right| &< \frac{1}{q^2} \\ \frac{1}{qb} \leq \left|\frac{aq - pb}{bq}\right| &< \frac{1}{q^2}, \end{aligned}$$

where the second inequality holds since all a, b, p, q are integers. It follows that we must have $q < b$, and there are only finitely many rationals $\frac{p}{q}$ that satisfy the statement. □

Note that heuristically, this gives us a characterisation of irrationality: If we can approximate a number too well with rationals, then it is irrational. We now provide a different proof of Dirichlet's Theorem, using the properties of the Farey fractions we proved previously.

Lemma 3.5. *If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent elements in $\mathfrak{F}_Q$, then*

$$\left|\frac{a}{b} - \frac{a+c}{b+d}\right| \leq \frac{1}{b(Q+1)} \text{ and } \left|\frac{c}{d} - \frac{a+c}{b+d}\right| \leq \frac{1}{d(Q+1)}.$$

Proof. The proof is just a matter of symbol pushing.

$$\begin{aligned} \left|\frac{a}{b} - \frac{a+c}{b+d}\right| &\leq \frac{1}{b(Q+1)} \\ \iff \left|\frac{ab + ad - ab - bc}{b(b+d)}\right| &\leq \frac{1}{b(Q+1)} \\ \iff \left|\frac{1}{b+d}\right| &\leq \frac{1}{(Q+1)} \end{aligned}$$

which is true, since $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent elements in $\mathfrak{F}_Q$(If not, a quick check shows that they cannot be adjacent elements). The other inequality can be proven similarly. □

We use this lemma to prove Dirichlet's Theorem.

Alternate proof to Theorem 3.3. We fix a positive integer Q. WLOG, suppose that $\alpha \in [0,1]$. Let $\frac{a}{b} < \alpha < \frac{c}{d}$, where $\frac{a}{b}, \frac{c}{d}$ are consecutive elements of $\mathfrak{F}_Q$. Suppose $\alpha < \frac{a+c}{b+d}$. It follows that

$$\left|\frac{a}{b} - \alpha\right| < \left|\frac{a}{b} - \frac{a+c}{b+d}\right| \leq \frac{1}{b(Q+1)}.$$

The $\alpha < \frac{a+c}{b+d}$ case can be proven similarly; this concludes the proof. For the infiniteness of rationals satisfying the inequality given that α is irrational, note that

$$\left|\alpha - \frac{a}{b}\right| < \left|\frac{a}{b} - \frac{c}{d}\right| = \frac{1}{bd} < \frac{1}{Q}.$$

Therefore, as we consider Farey Sequences of higher order, we cannot have the same fraction being a good approximation to α, since that would imply that $\frac{a}{b} = \alpha$, which is a contradiction. So it follows that we can repeat the above argument infinitely many times. Hence, we conclude that there are infinitely many fractions that satisfy the theorem. □

One might ask: is the inequality in Theorem 3.4 the best bound we could do? That is, could we replace the upper bound with a smaller quantity? Could we increase the exponent of q?[5] What about adding a constant in front of q^2? In 1891, Hurwitz discovered the best possible constant.

3.2 Hurwitz's Theorem

We first start with a lemma.

Lemma 3.6. *If x and y are positive integers, then at most one of the following inequalities can hold:*

$$\frac{1}{xy} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{y^2}\right), \qquad \frac{1}{x(x+y)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{(x+y)^2}\right).$$

Proof. Suppose that both inequalities hold. Then we have both

$$\sqrt{5}xy \geq y^2 + x^2, \qquad (x)(x+y)\sqrt{5} \geq (x+y)^2 + x^2.$$

So

$$\begin{aligned} xy\sqrt{5} + (x)(x+y)\sqrt{5} &\geq y^2 + x^2 + (x+y)^2 + x^2 \\ 0 &\geq 2y^2 + 2xy - 2xy\sqrt{5} + 3x^2 - \sqrt{5}x^2 \\ 0 &\geq 4y^2 - 4xy(\sqrt{5}-1) + (6-2\sqrt{5})x^2 \\ 0 &\geq (2y - (\sqrt{5}-1)x)^2, \end{aligned}$$

which is false since x, y are positive integers. □

Theorem 3.7 (Hurwitz, 1891)**.** *Let α be an irrational number. Then there exists infinitely many rational numbers $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Proof. WLOG, suppose $\alpha \in (0,1)$. Fix N, where N is some positive integer. Choose two consecutive elements of $\mathfrak{F}_N$, say $\frac{a}{b}$ and $\frac{c}{d}$ with $\frac{a}{b} < \alpha < \frac{c}{d}$. What is N? Now we show that one of $\frac{a}{b}, \frac{c}{d}, \frac{a+c}{b+d}$ satisfies the claim. Suppose not. Let $\alpha < \frac{a+c}{b+d}$. Then we have the following three inequalities:

$$\left|\frac{a+c}{b+d} - \alpha\right| \geq \frac{1}{\sqrt{5}(b+d)^2} \tag{2}$$

$$\left|\alpha - \frac{a}{b}\right| \geq \frac{1}{\sqrt{5}b^2} \tag{3}$$

$$\left|\frac{c}{d} - \alpha\right| \geq \frac{1}{\sqrt{5}d^2}. \tag{4}$$

After removing the absolute signs, by inequality (2) and (3):

$$\frac{a+c}{b+d} - \frac{a}{b} \geq \frac{1}{\sqrt{5}(b+d)^2} + \frac{1}{\sqrt{5}b^2}.$$

So

$$\frac{1}{(b)(b+d)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{(b+d)^2} + \frac{1}{b^2}\right).$$

But also, using inequality (3) and (4),

$$\begin{aligned} &\frac{c}{d} - \frac{a}{b} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b^2} + \frac{1}{d^2}\right) \\ \iff &\frac{1}{b(b+d)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b^2} + \frac{1}{d^2}\right). \end{aligned}$$

[5] We will answer this in Chapter xxx

So we have a contradiction to Lemma 3.6. Now we show that there exists infinitely many rationals that satisfy this inequality. Note that

$$\left|\alpha - \frac{a}{b}\right| < \left|\frac{a}{b} - \frac{c}{d}\right| = \frac{1}{bd} < \frac{1}{N},$$

where N is the order of the Farey Sequence. To prove that there are infinitely many rationals that satisfy the inequality, we use the same approach as in the alternate proof of Dirichlet's Theorem. The proof is similar for the case $\alpha > \frac{a+c}{b+d}$, and this concludes the proof. □

Next, we will prove that the constant $\sqrt{5}$ is the best possible choice in that the result would no longer hold if we choose a constant larger than $\sqrt{5}$. We start with a lemma.

Lemma 3.8.

(a) *Show that for any rational $\frac{a}{b}$, it follows that*

$$\left|\frac{a}{b} - \varphi\right| \left|\frac{a}{b} - \overline{\varphi}\right| = \frac{1}{b^2}\left|a^2 - ab - b^2\right|.$$

(b) *Show that $a^2 - ab - b^2$ cannot equal zero (recall that $b \neq 0$.)*

(c) *Deduce that*

$$\left|\frac{a}{b} - \varphi\right| \left|\frac{a}{b} - \overline{\varphi}\right| \geq \frac{1}{b^2}.$$

(d) *If for some positive real number m, $\left|\frac{a}{b} - \varphi\right| < \frac{1}{mb^2}$, then $\left|\frac{a}{b} - \overline{\varphi}\right| < \frac{1}{mb^2} + \sqrt{5}$.*

Proof.

(a) Expanding the expression:

$$\begin{aligned}\left|\frac{a}{b} - \varphi\right| \left|\frac{a}{b} - \overline{\varphi}\right| &= \frac{1}{b^2}\left|a^2 - ab\varphi - ab\overline{\varphi} + \varphi\overline{\varphi}b^2\right| \\ &= \frac{1}{b^2}\left|a^2 - ab - b^2\right|\end{aligned}$$

(b) Suppose that $\frac{a}{b}$ are in lowest terms. Then $\gcd(a, b) = 1$. If $a^2 - ab - b^2 = 0$, then

$$\frac{a^2}{b} = 1 + a.$$

Since $1 + a$ is an integer, we have $a \mid b$, a contradiction.

(c) Follows from (b).

(d) Using the triangle inequality,

$$\begin{aligned}\left|\frac{a}{b} - \overline{\varphi}\right| = \left|\frac{a}{b} - \varphi + \varphi - \overline{\varphi}\right| &\leq \left|\frac{a}{b} - \varphi\right| + |\varphi - \overline{\varphi}| \\ &\leq \frac{1}{mb^2} + \sqrt{5}\end{aligned}$$

□

We now show that the constant in Hurwitz's Theorem is the best we can do.

Theorem 3.9. *The constant $\sqrt{5}$ in Hurwitz's Theorem is best possible. That is, Theorem 3.7 does not hold if $\sqrt{5}$ is replaced by a larger value*

Proof. Let $\varphi = \frac{1+\sqrt{5}}{2}$, and suppose that for this φ, there exists a real positive number m, such that

$$\left|\varphi - \frac{a_i}{b_i}\right| < \frac{1}{m{b_i}^2},$$

for infinitely many distinct fractions $\frac{a_i}{b_i}$. It follows that $b_i \to \infty$ as $i \to \infty$. Suppose not, then we have that b_i is bounded. Since there are infinitely many fractions that satisfy the inequality, a_i is unbounded, which will give us a contradiction. From Lemma 3.8, for any $\frac{a_i}{b_i}$ we have

$$\left|\frac{a_i}{b_i} - \varphi\right|\left|\frac{a_i}{b_i} - \overline{\varphi}\right| \geq \frac{1}{{b_i}^2},$$

and

$$\left|\frac{a_i}{b_i} - \overline{\varphi}\right| < \frac{1}{m{b_i}^2} + \sqrt{5}.$$

So we have

$$\frac{1}{m{b_i}^2} \leq \left|\frac{a_i}{b_i} - \varphi\right|\left|\frac{a_i}{b_i} - \overline{\varphi}\right| < \left(\frac{1}{m{b_i}^2}\right)\left(\frac{1}{m{b_i}^2} + \sqrt{5}\right).$$

It follows that

$$1 \leq \frac{\sqrt{5}}{m} + \frac{1}{m^2{b_i}^2}.$$

As $i \to \infty$, $b_i \to \infty$, so

$$1 \leq \frac{\sqrt{5}}{m}, \qquad \text{and} \qquad m \leq \sqrt{5}.$$

This concludes the proof. □

The proof of Theorem 3.9 reveals that we cannot do much better than the constant $\sqrt{5}$, since the smaller upper bound will no longer hold for $\alpha = \frac{1+\sqrt{5}}{2}$.. However, there are still some questions we can ask: If we exclude $\frac{1+\sqrt{5}}{2}$, can we get a better constant? Also, how does one even compute the rational approximates for any real number α? We answer the latter question in the next section of this article via continued fractions. The former question will be dealt with in Chapter 6.

4 The Theory of Continued Fractions

We know that by Dirichlet's Theorem, there exists a rational number $\frac{p}{q}$ such that $1 \leq q \leq Q$ and

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q(Q+1)}.$$

We now aim to compute the fraction $\frac{p}{q}$, by developing an algorithm.

Lemma 4.1. *Let $x_0, x_1, x_2, \ldots$ be real numbers such that $x_n \neq 0$ for $n > 0$. If we define the numbers p_n and q_n by*

$$\begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}, \tag{5}$$

then

$$\frac{p_n}{q_n} = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}}}.$$

Proof. We first check that p_n and q_n is in fact, well defined. By definition,

$$\begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Now note that we can also define p_n and q_n with the expression:

$$\begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_{n+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix}.$$

Using the first expression, we can write

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}\begin{pmatrix} x_{n+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix},$$

where the second column entries of the product on the left is

$$\begin{pmatrix} \cdots & p_n \\ \cdots & q_n \end{pmatrix}.$$

So the $p_n's$ and $q_n's$ are in fact well defined.

We now proceed to prove the main result by induction. When $n = 1$,

$$\begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_0x_1 + 1 & x_0 \\ x_1 & 1 \end{pmatrix} = \begin{pmatrix} p_1 & p_0 \\ q_1 & q_0 \end{pmatrix}.$$

Since

$$\frac{p_1}{q_1} = x_0 + \frac{1}{x_1} = \frac{x_0x_1 + 1}{x_1},$$

the statement is true for $n = 1$.

Suppose that the statement is true for real numbers $y_0, y_1, \ldots, y_{n-1}$. In other words, if

$$\begin{pmatrix} y_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} y_{n-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p'_{n-1} & p'_{n-2} \\ q'_{n-1} & q'_{n-2} \end{pmatrix},$$

then

$$\frac{p'_{n-1}}{q'_{n-1}} = y_0 + \cfrac{1}{y_1 + \cfrac{1}{\ddots + \cfrac{1}{y_{n-1}}}}.$$

Suppose we have the sequence of $n-1$ real numbers $x_0, x_1, \ldots, x_n$, and we define p_n and q_n by:

$$\begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}. \tag{6}$$

Now set $y_0 = x_0, y_1 = x_1, \ldots y_{n-1} = \frac{x_{n-1}x_n+1}{x_n}$. Then by the induction hypothesis, we have

$$\begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \frac{x_{n-1}x_n+1}{x_n} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p'_{n-1} & p'_{n-2} \\ q'_{n-1} & q'_{n-2} \end{pmatrix}. \tag{7}$$

Using equations 6 and 7,

$$\begin{aligned}\begin{pmatrix} p'_{n-1} & p'_{n-2} \\ q'_{n-1} & q'_{n-2} \end{pmatrix} &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}\begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} x_{n-1} & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} \frac{x_{n-1}x_n+1}{x_n} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{p_n}{x_n} & p_n \\ \frac{q_n}{x_n} & -x_nq_n \end{pmatrix}.\end{aligned}$$

Hence,

$$p'_{n-1} = \frac{p_n}{x_n}, \qquad \text{and} \qquad q'_{n-1} = \frac{q_n}{x_n}.$$

So finally,

$$\begin{aligned}\frac{p_n}{q_n} &= \frac{p'_{n-1}}{q'_{n-1}} \\ &= x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{x_n x_{n-1} + 1}{x_n}}} \\ &= x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_{n-1} + \cfrac{1}{x_n}}}},\end{aligned}$$

which concludes the proof. □

For ease of notation, we write

$$[x_0, x_1, \dots, x_N]$$

to denote the expression

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \cfrac{1}{x_N}}}}.$$

The next corollary gives a recurrence relation that can be used to easily generate the p_n's and q_n's.

Corollary 4.2. *Given the notation in Lemma 4.1, it follows that*

$$p_{N+1} = x_{N+1}p_N + p_{N-1} \qquad \textit{and} \qquad q_{N+1} = x_{n+1}q_N + q_{N-1}$$

for all $N \geq 1$.

Proof. The proof is again by induction. Setting $n = 1$, in Lemma 4.1:

$$\begin{aligned}\begin{pmatrix} p_2 & p_1 \\ q_2 & q_1 \end{pmatrix} &= \begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_0x_1x_2 + x_2 + x_0 & x_0x_1 + 1 \\ x_1x_2 + 1 & x_1 \end{pmatrix}\end{aligned}$$

Hence

$$p_2 = x_0x_1x_2 + x_2 + x_0 = x_2p_1 + p_0, \qquad \text{and} \qquad q_2 = x_1x_2 + 1 = x_2q_1 + q_0.$$

Suppose that the statement holds for $N = n$, then again by Lemma 4.1,

$$\begin{aligned}\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} &= \begin{pmatrix} x_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}\begin{pmatrix} x_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_nx_{n+1} + p_{n-1} & p_n \\ q_nx_{n+1} + q_{n-1} & q_n \end{pmatrix}\end{aligned}$$

So $p_{n+1} = p_nx_{n+1} + p_{n-1}$, and $q_{n+1} = q_nx_{n+1} + q_{n-1}$. This concludes the proof. □

We now see that the $p_n's$ and $q_n's$ are related in a simple way.

Corollary 4.3. *For all $N \geq 1$, $p_N q_{N-1} - p_{N-1} q_N = (-1)^{N+1}$.*

Proof. Take determinants on both sides in Lemma 4.1. □

The quantities $\frac{p_N}{q_N}$ can be expressed just in terms of x_0 and the $q_n's$.

Lemma 4.4. *Given the notation in Lemma 4.1, for any integer $N \geq 0$,*

$$\frac{p_N}{q_N} = x_0 + \sum_{n=1}^{N} \frac{(-1)^{n-1}}{q_n q_{n-1}}.$$

Proof. Yet again another proof by induction. When $N = 1$, by Lemma 4.1

$$\frac{p_1}{q_1} = x_0 + \frac{1}{x_1} = x_0 + \frac{1}{q_0 q_1}.$$

Suppose that the statement holds for N-1. Using Corollary 4.3,

$$\frac{p_N}{q_N} = \frac{(-1)^{N+1}}{q_N q_{N-1}} + \frac{p_{N-1}}{q_{N-1}}.$$

Applying the inductive hypothesis,

$$\begin{aligned}
\frac{p_N}{q_N} &= \frac{(-1)^{N+1}}{q_N q_{N-1}} + \frac{p_{N-1}}{q_{N-1}}. \\
&= x_0 + \sum_{n=1}^{N-1} \frac{(-1)^{n-1}}{q_n q_{n-1}} + \frac{(-1)^{N+1}}{q_N q_{N-1}} \\
&= x_0 + \sum_{n=1}^{N} \frac{(-1)^{n-1}}{q_n q_{n-1}}.
\end{aligned}$$

□

If we further assume that the real numbers x_n are positive, then we see an interesting growth pattern in the sequence $\{\frac{p_n}{q_n}\}$.

Corollary 4.5. *If $x_i > 0$ for all $i > 0$, then*

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_{2n}}{q_{2n}} < \cdots \qquad \cdots < \frac{p_{2m+1}}{q_{2m+1}} < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Remark. This inequality encodes three different conclusions: the even-indexed terms are strictly decreasing, the odd-indexed terms are strictly increasing, and every even-indexed term is less than every odd-indexed term.

Proof. [HW09, p.168-169] To prove that the even-indexed terms are strictly increasing, using our prveious lemma, we consider the difference

$$\begin{aligned}
\frac{p_{2n+2}}{q_{2n+2}} - \frac{p_{2n}}{q_{2n}} &= \left(x_0 + \sum_{n=1}^{2n+2} \frac{(-1)^{n-1}}{q_n q_{n-1}} \right) - \left(x_0 + \sum_{n=1}^{2n} \frac{(-1)^{n-1}}{q_n q_{n-1}} \right) \\
&= -\frac{1}{q_{2n+1} q_{2n}} + \frac{1}{q_{2n+1} q_{2n+2}} \\
&> 0,
\end{aligned}$$

where the last inequality follows from the fact that $x_i > 0$ for all i. The proof that the odd-indexed terms are strictly decreasing is similar, again we consider

$$\begin{aligned}\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n-1}}{q_{2n-1}} &= \left(x_0 + \sum_{n=1}^{2n+1} \frac{(-1)^{n-1}}{q_n q_{n-1}}\right) - \left(x_0 + \sum_{n=1}^{2n-1} \frac{(-1)^{n-1}}{q_n q_{n-1}}\right) \\ &= \frac{1}{q_{2n+1}q_{2n}} - \frac{1}{q_{2n}q_{2n-1}} \\ &< 0,\end{aligned}$$

and we are done. Now we prove that all even-indexed terms are less than every odd-indexed term. Suppose that we have an arbitrary even-indexed term, $\frac{p_{2m}}{q_{2m}}$, and another arbitrary odd-indexed term, $\frac{p_{2n+1}}{q_{2n+1}}$. We consider three different cases:

1. If $n = m$, we can directly take the difference and use the previous lemma, which gives us

$$\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}} = \frac{1}{q_{2n+1}q_{2n}},$$

   which is positive, hence $\frac{p_{2n+1}}{q_{2n+1}} > \frac{p_{2n}}{q_{2n}}$.

2. If $n < m$, since the odd-indexed terms are strictly decreasing, we have

$$\frac{p_{2m+1}}{q_{2m+1}} - \frac{p_{2n}}{q_{2n}} > \frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}} > 0.$$

3. If $n > m$, then since the even-indexed terms are strictly increasing,

$$\frac{p_{2m+1}}{q_{2m+1}} - \frac{p_{2n}}{q_{2n}} > \frac{p_{2m+1}}{q_{2m+1}} - \frac{p_{2m}}{q_{2m}} > 0.$$

This concludes the proof. □

We now restrict the $x_n's$ to strictly be integers.

Theorem 4.6. *If $a_0, a_1, \ldots$ are integers with $a_n > 0$ for all $n > 0$, then*

$$\lim_{N\to\infty} [a_0, a_1, \ldots, a_N] = \alpha$$

for some irrational α. Moreoever, for all $n \geq 0$,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

Proof. [Lon87, p.221-222] The first statement is equivalent to showing that

$$\frac{p_n}{q_n} \to \alpha, \quad \text{as} \quad N \to \infty.$$

From Corollary 4.5, we know that the even indexed terms are increasing and bounded above, and that the odd-indexed terms are decreasing and bounded below. Hence, it follows that both sequences converge to a limit. Let

$$\lim_{N\to\infty} \frac{p_{2n}}{q_{2n}} = S, \qquad \lim_{N\to\infty} \frac{p_{2n+1}}{q_{2n+1}} = T.$$

It remains to show that $S = T$. Observe that

$$\left|\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}}\right| = \frac{1}{q_{2n}q_{2n+1}},$$

taking limits on both sides as $N \to \infty$, we can conclude that

$$S = \lim_{N\to\infty} \frac{p_{2n}}{q_{2n}} = \lim_{N\to\infty} \frac{p_{2n+1}}{q_{2n+1}} = T,$$

as required, since the $q_n's$ are increasing. We now check that α satisfies the inequality. Indeed, since α is the common limit of both the odd and even-indexed sequences, α must be between any n and $n+1$. Hence,

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}}.$$

We now use the inequality to show that given that the inequality holds for all n, α must be irrational. Suppose not, write $\alpha = \frac{a}{b}$. The inequality can now be rewritten as

$$\left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$
$$|aq_n - p_n b| < \frac{b}{q_{n+1}},$$

which we claim can only hold for finitely many n. In fact, since the $q_n's$ are increasing, there exists an integer m such that $q_N > b$ for all $m > N$. Therefore, $|aq_n - p_n b| < 1$, and since all a, b, p_n, q_n are integers, it will lead to a contradiction to Corollary 4.5. □

The expression $[a_0, a_1, \dots]$ from Theorem 4.6 is called the *simple continued fraction expression of* α. The integer a_n is called the *nth partial quotient of* α, the rational number $\frac{p_n}{q_n}$ is the *nth convergent of* α.

We now describe an algorithm for determining the continued fraction expansion for any given real α. Let α be a real number. We write $\lfloor \alpha \rfloor$ to denote the integer part of α and $\{\alpha\}$ to denote the fractional part of α.

Suppose we are given an α and wish to find its continued fraction expansion. Let's consider the following procedure: Define $\alpha_0 = \alpha$ and let $a_0 = \lfloor \alpha \rfloor$. If $a_0 \notin \mathbb{Z}$, then let $\alpha_1 = 1/(\alpha_0 - a_0)$. In general, if α_n has been defined, then let $a_n = \lfloor \alpha_n \rfloor$, and if $\alpha_n \notin \mathbb{Z}$, then let $\alpha_{n+1} = 1/(\alpha_n - a_n)$.

Proposition 4.7. *Let the $a_n's$ be as defined in the previous algorithm.*

(a) Show that $\alpha = [a_0, a_1, \dots]$ and that $a_n \in \mathbb{Z}$ and for all $n > 0, a_n > 0$.

(b) Show that p_n and q_n are integers and that $q_1, q_2, \dots$ is an increasing sequence of positive integers. Also show that the integers p_n and q_n are relatively prime.

Proof. (a) By Theorem 4.6, we know that the limit exists, say the simple continued fraction expansion converges to β. Now we have to show that in fact $\alpha = \beta$.

$$|\alpha - \beta| \le \left|\alpha - \frac{p_n}{q_n}\right| + \left|\beta - \frac{p_n}{q_n}\right|$$
$$< \left|\alpha - \frac{p_n}{q_n}\right| + \frac{\varepsilon}{2},$$

by Theorem 4.6. So it remains to show that for every $\varepsilon > 0$, there exists an $N > 0$ such that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{\varepsilon}{2},$$

for all $n > N$. Indeed, we start by writing α in the simple continued fraction expansion form (which could be infinite). We then directly simplify the expression. Therefore,

$$\left|\alpha - \frac{p_n}{q_n}\right| = \left|\frac{\alpha_n - a_n}{\Pi_{i=1}^n [a_i, a_{i-1}, \dots, a_n][a_i, a_{i-1}, \dots \alpha_n]}\right|.$$

Now, since each term on the denominator is greater than 1 (by definition of a_n), and that $|\alpha_n - a_n| < 1$, for small $\varepsilon > 0$, we choose N such that

$$\left|\frac{\alpha_N - a_N}{\Pi_{i=1}^N [a_i, a_{i-1}, \dots, a_N][a_i, a_{i-1}, \dots \alpha_N]}\right| < \frac{\varepsilon}{2}.$$

So, for all $n > N$, we have

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{\varepsilon}{2},$$

as required.

To show that for all $n > 0$, $a_n > 0$, note that $\alpha_n - a_n > 0$ for all $n > 0$, from which the claim follows.

(b) Since all $a_n^{\prime}s$ are integers by construction, inductively, using Corollary 4.2, it follows that p_n and q_n are integers. To show that $q_1, q_2, \ldots$ is an increasing sequence of positive integers, we note that $q_0 = 1 > 0$, and $q_1 = a_1 > 0$. So again by Corollary 4.2, the sequence of $q_n^{\prime}s$ is increasing.

The result that the integers p_n and q_n are relatively prime follows from Corollary 4.3.

□

We now have a new criterion for rationality.

Theorem 4.8. *The real number α is rational if and only if the process described above terminates in finitely many steps.*

Proof. [Old63, p.14] We start with the only if direction. Let $\alpha = \frac{a}{b}$, with $b > 0$. By the algorithm $\frac{a}{b} = \alpha = a_0 + \frac{1}{\alpha_1}$, so $a = ba_0 + \frac{b}{\alpha_1}$. We deduce that $\frac{b}{\alpha_1} \in \mathbb{Z}$. Now we define $\frac{b}{\alpha_i} = b_i$. So

$$\frac{b}{b_1} = \alpha_1 = \lfloor \alpha_1 \rfloor + \{\alpha_1\} = a_1 + \frac{1}{\alpha_2}.$$

Multiplying by b_1,

$$b = b_1\alpha_1 + \frac{b_1}{\alpha_2}.$$

Inductively, we get the system of equations:

$$\begin{aligned} b &= a_1 b_1 + b_2 \\ b_1 &= a_2 b_2 + b_3 \\ &\vdots \end{aligned}$$

Notice that $b > b_1 > b_2 > \cdots > 0$, since all of the $b_i^{\prime}s$ are positive. Therefore, there must exist some N with $b_{N+2} = 0$. If $b_{N+2} = 0$, then $b_i = a_{i+1}b_{i+1}$. Since $\frac{b_i}{b_{i+1}} = \alpha_{i+1} = a_{i+1}$, it follows that $\alpha_{i+1} \in \mathbb{Z}$ and the algorithm terminates.

For the if direction, we just have to slowly expand the fraction from the bottom, from which the claim follows. □

We now answer the question proposed in the previous chapter: how does one compute the fractions that approximate any real number α, and how close of an approximation it is?

Theorem 4.9. *Suppose that $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ are the convergents associated with the irrational number α. Let $Q > 1$ be an integer. If n is the index satisfying $q_n \leq Q < q_{n+1}$, then*

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{(Q+1)q_n}.$$

Proof. Since Q, q_n, q_{n+1} are all integers, if $Q < q_{n+1}$, then $Q + 1 \leq q_{n+1}$. Combining this with Theorem 4.6, the claim follows. □

Thus, we see that the convergents are rational numbers that satisfy Dirichlet's Theorem. In the next chapter, we will show that the convergents are, in some sense, the "best" rational approximation to α.

5 Enforcing the law of best approximates.

From Theorem 3.3, we can deduce that the inequality is equivalent to

$$|\alpha q - p| \leq \frac{1}{Q+1}.$$

Therefore, we see that the integer p is easily determined by q: p is the nearest integer to αq. Thus, we define a new function to simplify our notation.

Definition 5.1. We define the *distance to the nearest integer function*, $\| \quad \|$, by,

$$\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}.$$

Dirichlet's Theorem can now be restated as: There exists an integer q satisfying $1 \leq q \leq Q$ such that $\|\alpha q\| \leq (Q+1)^{-1}$.

In this section of the article, we ask: for an irrational number α, is there an algorithm to find the complete list of increasing positive integers $\{q_1, q_2, \dots\}$ such that

$$\|\alpha q_1\| > \|\alpha q_2\| > \cdots > \|\alpha q_n\| > \dots \tag{8}$$

We note that $\|\alpha q\| = q\left|\alpha - \frac{p}{q}\right|$, so in fact we are not only asking to find rationals that approach α but also to have the feature that their distances from α are getting smaller as compared to the size of their denominators. In other words, we are finding the complete list of what are known as *the best approximates to α.*

We will soon discover that the convergents given by the continued fraction expansion of α are, in fact, a complete collection of best approximations to α. This result, due to Lagrange, is generally known as *The Law of Best Approximates.*

We begin by considering some basic properties of the "distance to nearest integer function".

Proposition 5.2. *Let $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{Z}$. Prove that $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ and $\|n\alpha\| \leq |n|\|\alpha\|$. The first inequality is known as the triangle inequality.*

Proof. Let k and k' denote the closest integer to α and β respectively. Then

$$\begin{aligned} \|\alpha\| + \|\beta\| =& |\alpha - k| + |\beta - k'| \\ \geq& |\alpha + \beta - k - k'| \\ \geq& \min\{|\alpha + \beta - n| : n \in \mathbb{Z}\} \\ =& \|\alpha + \beta\|. \end{aligned}$$

Whereby the second inequality is the usual triangle inequality, and the third inequality holds because we are considering the minimum distance across all possible $n \in \mathbb{Z}$.

To prove the second inequality, we use a similar idea. Let k be the closest integer to α. Then,

$$\begin{aligned} |n|\|\alpha\| =& |n||k - \alpha| \\ =& |nk - n\alpha| \\ \geq& \min\{|n\alpha - m| : m \in \mathbb{Z}\}. \end{aligned}$$

□

We can write a formula for $\|\alpha\|$ in terms of the fractional part of α.

$$\|\alpha\| = \begin{cases} \{\alpha\}, & \text{if } \{\alpha\} \leq \frac{1}{2} \\ 1 - \{\alpha\}, & \text{if } \{\alpha\} > \frac{1}{2} \end{cases}$$

.

Before finding the best approximates, we introduce a few more important identities and definitions.

Definition 5.3. Let α be an irrational number, with $\alpha = [a_0, a_1, \dots]$. We define the *Nth complete quotient* by $\alpha_N = [a_N, a_{N+1}, \dots]$.

Lemma 5.4. *For an irrational number α having convergents $\frac{p_n}{q_n}$ and complete quotients α_n, it follows that*

$$\alpha = \frac{\alpha_{N+1}p_N + p_{N-1}}{\alpha_{N+1}q_N + q_{N-1}}.$$

Proof. Note that the continued fraction of α can be rewritten as $\alpha = [a_0, a_1, \dots] = [a_0, a_1, \dots, a_N, \alpha_{N+1}]$. Using Lemma 4.1, we first define p'_{n+1} and q'_{n+1} by

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{N+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p'_{N+1} & p'_N \\ q'_{N+1} & q'_N \end{pmatrix}. \tag{9}$$

Now we note that using our usual definition of p_n and q_n, viz

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_N & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1} \end{pmatrix},$$

we rewrite equation 9

$$\begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1} \end{pmatrix} \begin{pmatrix} \alpha_{N+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p'_{N+1} & p'_N \\ q'_{N+1} & q'_N \end{pmatrix}.$$

Hence, after noting that $\frac{p'_{N+1}}{q'_{N+1}} = [a_0, a_1, \dots a_N, \alpha_{N+1}] = \alpha$, multiplying out the previous matrix gives us

$$\begin{pmatrix} \alpha_{N+1}p_N + p_{N-1} & p_N \\ \alpha_{N+1}q_N + q_{N-1} & q_N \end{pmatrix} = \begin{pmatrix} p'_{N+1} & p'_N \\ q'_{N+1} & q'_N \end{pmatrix},$$

and the conclusion follows. □

Lemma 5.5. *For an irrational number α having convergents $\frac{p_n}{q_n}$ and complete quotients, α_n, it follows that*

$$q_N\alpha - p_N = \frac{(-1)^N}{\alpha_{N+1}q_N + q_{N-1}}.$$

Proof. Write

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{N+1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_{N+1}p_N + p_{N-1} & p_N \\ \alpha_{N+1}q_N + q_{N-1} & q_N \end{pmatrix}.$$

Now taking determinants on both sides,

$$(-1)^{N+1} = (\alpha_{N+1}p_N + p_{N-1})q_N - (\alpha_{N+1}q_N + q_{N-1})p_N,$$

simplifying and factorising the expression, we get

$$q_N\alpha - p_N = \frac{(-1)^N}{\alpha_{N+1}q_N + q_{N-1}},$$

and we are done. □

Lemma 5.6. *Given the notation in Lemma 5.5, for all $N > 0$,*

$$|\alpha q_N - p_N| < \frac{1}{q_N + q_{N-1}} < |\alpha q_{N-1} - p_{N-1}|.$$

Proof. From Lemma 5.5,

$$\begin{aligned} |\alpha q_N - p_N| &= \frac{1}{\alpha_{N+1}q_N + q_{N-1}} \\ &< \frac{1}{q_N + q_{N-1}}. \end{aligned}$$

For the other inequality, using Lemma 5.5 along with Lemma 5.4,

$$\begin{aligned} |\alpha q_{N-1} - p_{N-1}| &= \left| q_{N-1} \frac{\alpha_{N+1} p_N + p_{N-1}}{\alpha_{N+1} q_N + q_{N-1}} - p_{N-1} \right| \\ &= \frac{\alpha_{N+1}}{\alpha_{N+1} q_N + q_{N-1}} \\ &= \frac{1}{q_N + \frac{q_{N-1}}{\alpha_{N+1}}} \\ &> \frac{1}{q_N + q_{N-1}}, \end{aligned}$$

as required. ☐

Theorem 5.7. *Let α be an irrational real number having convergents $\frac{p_n}{q_n}$. Then*

$$|\alpha q_0 - p_0| > |\alpha q_1 - p_1| > \cdots > |\alpha q_n - p_n| > \ldots.$$

Proof. This follows immediately from the previous lemma. ☐

We will now show that the denominators of the convergents satisfy the string of inequalities in (8).

Corollary 5.8. *Given the notation in Theorem 5.7 it follows that*

$$\|\alpha q_1\| > \|\alpha q_2\| > \cdots > \|\alpha q_n\| > \ldots.$$

Proof. We just have to show that p_n is in fact the closest integer to αq_n for all n, Indeed, by Theorem 4.9,

$$|q_n \alpha - p_n| < \frac{1}{2},$$

since $Q > 1$. This inequality holds for all n, and we are done. ☐

The denominators of the convergents satisfy the inequalities in (8), but do they form a *complete* list of integers satisfying (8)? In other words, does there exist an integer q with $q_{N-1} < q < q_N$ such that

$$\|\alpha q_{N-1}\| > \|\alpha q\| > \|\alpha q_N\|.$$

We will show that the answer to this question is no. Thus, the denominators of the convergents are indeed a complete list of integers satisfying (8). We begin with a gentle lemma.

Lemma 5.9. *Let α be an irrational number, and let p, q be two integers. There exists a unique integer solution x, y to the system*

$$\begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}.$$

Proof. Note that

$$\det \begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1} \end{pmatrix} = (-1)^{N+1}.$$

Now taking the inverse of the matrix above on both sides,

$$\begin{pmatrix} x \\ y \end{pmatrix} = (-1)^{N+1} \begin{pmatrix} q_{N-1} & -p_{N-1} \\ -q_N & p_N \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Since $p_N, q_N, p_{N-1}, q_{N-1}$ are all integers, the claim follows. ☐

We have finished all the buildup necessary to prove a theorem due to Lagrange from 1770, known as the *The Law of Best Approximates.*

Theorem 5.10. *Let α be an irrational real number, and let $\frac{p_N}{q_N}$ denote the Nth convergent of α. Suppose that $\frac{p}{q}$ is a rational number satisfying $1 \leq q \leq q_N$ and the ordered pair $(p,q) \neq (p_{N-1}, q_{N-1})$, $(p,q) \neq (p_N, q_N)$. Then*

1. $|\alpha q_0 - p_0| > |\alpha q_1 - p_1| > \cdots > |\alpha q_n - p_n| > \ldots$,
2. $|\alpha q - p| > |\alpha q_{N-1} - p_{N-1}|$.

Proof. 1. This is precisely Theorem 5.7.

2. By Lemma 5.9, we know that there exists integers x, y such that

$$p_N x + p_{N-1} y = p \quad \textit{and} \quad q_N x + q_{N-1} y = q.$$

We now consider three cases : $xy < 0, xy > 0$, and $xy = 0$. We first note that the case $xy > 0$ immediately contradicts the initial assumption $q \leq q_N$. Moving on, suppose $xy = 0$. It follows that we must have $y \neq 0$. Suppose otherwise, then $q_N x = q$, and since x is an integer, we reach a contradiction.[6] Now, if $x = 0$, then $p_{N-1} y = p$ and $q_{N-1} y = q$. Note that $y \neq 1$, otherwise we will again produce a contradiction. Now consider

$$\begin{aligned} |\alpha q - p| &= |\alpha q_{N-1} y - p_{N-1} y| \\ &= |y| |\alpha q_{N-1} - p_{N-1}| \\ &> |\alpha q_{N-1} - p_{N-1}|, \end{aligned}$$

since $|y| > 1$, as the claim requires. Now we consider the case where $xy < 0$. Again, we consider the quantity

$$\begin{aligned} |\alpha q - p| &= |\alpha(x q_N + y q_{N-1}) - x p_N - y p_{N-1}| \\ &= |x(\alpha q_N - p_N) + y(\alpha q_{N-1} - p_{N-1})| \\ &\leq |x| |\alpha q_N - p_N| + |y| |\alpha q_{N-1} - p_{N-1}|, \end{aligned}$$

where the final inequality is derived from the triangle inequality. Since $xy < 0$, we note that x, y have different parities. Now, combining this observation with Corollary 4.1, we deduce that we must have equality in the triangle inequality, since both terms have the same parity. Now, finally, since we have equality in the final line, the claim follows.

□

Remark. The second part of Lagrange's Theorem can be interpreted as follows: there does not exist an integer with lower complexity/height than q_N, for any N, that approximates it better than q_{N-1}. This is precisely what we set out to prove in the first place.

It follows immediately from Theorem 4.6 that the convergents also satisfy $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$. One might wonder: Is the converse true? That is, if I have a rational $\frac{p}{q}$ that satisfies the previous inequality, does that imply that the rational is a convergent of α? Although the answer is no, if we tighten the upper bound, the answer becomes yes. This is a result first discovered by Legendre.

We start with a simple lemma.

Lemma 5.11. *Let a and b be two distinct real numbers. Then*

$$ab < \frac{1}{2}(a^2 + b^2)$$

Proof. Consider the inequality $(a - b)^2 > 0$. □

Theorem 5.12. *Let $\frac{p_{N-1}}{q_{N-1}}$ and $\frac{p_N}{q_N}$ be consecutive convergents to α. Then at least one of those rationals will satisfy*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

[6] Note that x must be a positive integer as well, otherwise we cannot have $1 \leq q$.

Proof. [Bor+14] (p.35) Suppose not, then we have the following inequalities

$$\left|\alpha - \frac{p_n}{q_N}\right| \geq \frac{1}{2q_N^2},$$

$$\left|\alpha - \frac{p_{N-1}}{q_{N-1}}\right| \geq \frac{1}{2q_{N-1}^2}.$$

Adding both inequalities, we get

$$\begin{aligned}\left|\alpha - \frac{p_n}{q_N}\right| + \left|\alpha - \frac{p_{N-1}}{q_{N-1}}\right| &\geq \frac{1}{2}\left(\frac{1}{q_N^2} + \frac{1}{q_{N-1}^2}\right) \\ &> \frac{1}{q_N q_{N-1}},\end{aligned}$$

by the previous lemma. Now, WLOG, suppose that N is even. Then, after carefully removing the absolute values, we observe that

$$\begin{aligned}\frac{p_{N-1}}{q_{N-1}} - \alpha + \alpha - \frac{p_N}{q_N} &> \frac{1}{q_N q_{N-1}} \\ \frac{p_{N-1}}{q_{N-1}} - \frac{p_N}{q_N} &> \frac{1}{q_N q_{N-1}} \\ p_{N-1}q_N - p_N q_{N-1} &> 1\end{aligned}$$

which is a contradiction to Corollary 4.3. □

Theorem 5.13 (Legendre)**.** *Suppose that p and q are relatively prime integers with $q > 0$ and*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Then $\frac{p}{q}$ is a convergent of α.

Proof. Choose N such that $q_N \leq q < q_{N+1}$. Note that we have the following two inequalities to work with

$$|\alpha q - p| < \frac{1}{2q} \quad \textit{and} \quad |\alpha q_N - p_N| < \frac{1}{2q_N}.$$

Now, consider the quantity $|pq_N - p_N q|$. Using the triangle inequality again

$$\begin{aligned}|pq_N - qp_N| &\leq |q_N||\alpha q - p| + |q||\alpha q_N - p_N| \\ &< |q||\alpha q - p| + |q||\alpha q - p| \\ &< |q|\left|\frac{1}{2q}\right| + |q|\left|\frac{1}{2q}\right| \\ &= 1,\end{aligned}$$

where the second inequality follows from our initial assumption that $q_N < q$ and Theorem 5.10. Since p, q, p_N, q_N are all integers, the bound implies that $pq_N - qp_N = 0$, and so $\frac{p}{q} = \frac{p_N}{q_N}$, as required. □

6 Markoff's Spectrum and numbers

We now return to the question at the end of Chapter 3: Suppose that we exclude $\frac{1+\sqrt{5}}{2}$ from our considerations, could we get a better constant than $\frac{1}{\sqrt{5}}$? It certainly seems plausible; perhaps we could create a decreasing sequence of constants as we continually restrict the allowable choices for α.

This decreasing sequence of best possible constants forms the beginning of the *Markoff spectrum*. In 1880, A.Markoff discovered a stunning result showing a deep connection between these constants and integer solutions to the diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

Roughly speaking, we wish to find small constants c such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{c}{q^2}$$

has infinitely many rational solutions $\frac{p}{q}$. It is clear to see that the previous inequality is equivalent to

$$q\|\alpha q\| \leq c.$$

It thus follows that the optimal constant for a particular α given by

$$\mu(\alpha) = \liminf_{q\to\infty} q\|\alpha q\|.$$

The value $\mu(\alpha)$ is often referred to as the *Markoff constant for* α. We begin with an exploration of the quantity $\mu(\alpha)$.

Lemma 6.1. *Let $\alpha = [a_0, a_1, a_2, \dots]$ be an irrational number, and $\frac{p_N}{q_N} = [a_0, a_1, \dots, a_N]$ denote its Nth convergent. Then for all $N > 0$,*

$$\frac{q_{N-1}}{q_N} = [0, a_N, a_{N-1}, \dots, a_1].$$

Proof. Yet another proof by induction. We know that

$$q_0 = 1 \quad \text{and} \quad q_1 = a_1,$$

so the base case is true. Using Corollary 4.2,

$$\frac{q_N}{q_{N+1}} = \frac{1}{a_N + \frac{q_{N-1}}{q_N}},$$

and the conclusion follows. □

Lemma 6.2. *Let $\alpha = [a_0, a_1, a_2 \dots]$ be an irrational number and $\frac{p_N}{q_N}$ be its Nth convergent. Then for all N¿0,*

$$q_N\|\alpha q_N\| = \left([a_{N+1}, a_{N+2}, a_{N+3} \dots] + [0, a_N, a_{N-1}, \dots, a_1]\right)^{-1}.$$

Proof. By Lemma 5.5, we deduce that

$$\begin{aligned} q_N\|\alpha q_N\| &= \frac{q_N}{\alpha_{N+1}q_N + q_{N-1}} \\ &= \frac{1}{\alpha_{N+1} + \frac{q_{N-1}}{q_N}} \\ &= \left([a_{N+1}, a_{N+2}, a_{N+3} \dots] + [0, a_N, a_{N-1}, \dots, a_1]\right)^{-1}, \end{aligned}$$

where the last line follows from the previous lemma and the definition of α_{N+1}. □

Lemma 6.3. *For an irrational number $\alpha = [a_0, a_1, a_2, \dots]$,*

$$\mu(\alpha) = \liminf_{N\to\infty} \left([a_{N+1}, a_{N+2}, a_{N+3} \dots] + [0, a_N, a_{N-1}, \dots, a_1]\right)^{-1}.$$

Proof. We first note that Theorem 3.7 implies Theorem 5.13. Since we know that the best possible constant is the one in Theorem 3.7, by the definition of $\mu(\alpha)$, $\mu(\alpha) \leq \frac{1}{\sqrt{5}}$, and the conclusion follows. □

Title - Adding fractions wrongly, an introduction to Diophantine Approximation

What happens if we define the addition of fractions in the 'wrong' way: $a/b + c/d = a + c/b + d$? Surprisingly, this 'wrong' definition opens the door to a beautiful part of number theory: Diophantine Approximation, the study of how well the rationals can approximate real numbers. In this talk, we will uncover the structure of these 'wrongly added' fractions, and use it to prove a few of the fundamental results in the field, namely Dirichlet's Theorem, which guarantees infinitely many 'good' rational approximations, and Hurwitz's Theorem, which gives us the best possible bound for these approximations. We will explore how Diophantine Approximation connects to transcendental number theory, proving Liouville's Theorem and constructing explicit examples of transcendental numbers, thereby establishing their existence. The talk will be accessible to all undergraduates.

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